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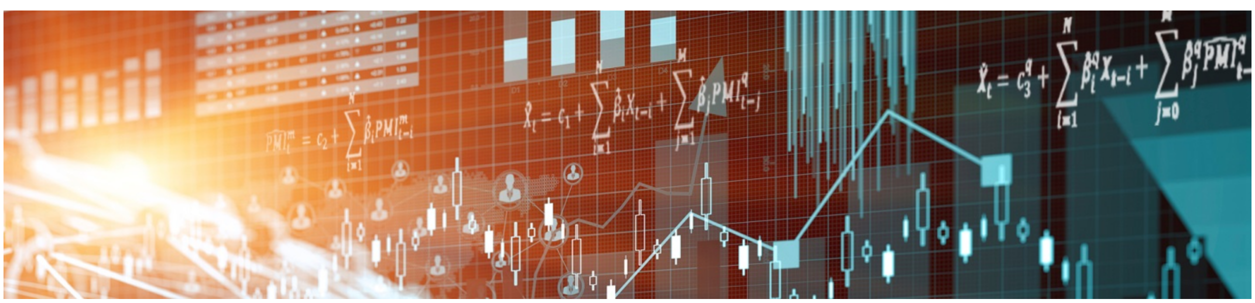
Model Uncertainty and Wealth Distribution



by **Edouard Djeutem and Shaofeng Xu**

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Model Uncertainty and Wealth Distribution

by

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Abstract

This paper studies the implications of model uncertainty for wealth distribution in a tractable general equilibrium model with a borrowing constraint and robustness *à la* Hansen and Sargent (2008). Households confront model uncertainty about the process driving the return of the risky asset, and they choose robust policies. We find that in the presence of a borrowing constraint, model distortion varies non-monotonically with wealth. Robustness generates two forces that amplify wealth inequality. On the one hand, it increases the speed at which the wealth of unlucky households hits the borrowing constraint. On the other hand, it leads richer households to invest a disproportionately larger share of wealth in the higher yielding asset. Our study also shows that model uncertainty results in an aggregate welfare loss unevenly distributed across households.

Bank topics: Economic Model, Business fluctuations and cycles, Asset pricing
JEL codes: D3; D8; E2

Résumé

Les auteurs étudient les implications de l'incertitude de modèle pour la distribution de la richesse dans un modèle d'équilibre général maniable intégrant une contrainte d'emprunt et une robustesse à la Hansen et Sargent (2008). Les ménages sont confrontés à l'incertitude de modèle liée au processus qui détermine le rendement de l'actif risqué, et prennent des décisions robustes. Les auteurs constatent qu'en présence d'une contrainte d'emprunt, la distorsion dans le modèle varie de façon non monotone en fonction de la richesse. La robustesse génère deux forces qui amplifient les inégalités de richesse : la première accroît la vitesse à laquelle les ménages moins bien nantis se heurtent à la contrainte d'emprunt, et la seconde fait que les ménages mieux nantis investissent une part disproportionnellement plus grande de leur richesse dans l'actif au rendement supérieur. L'étude montre également que l'incertitude de modèle donne lieu à une perte de bien-être généralisée et répartie inégalement entre les ménages.

Sujets : Modèle économique, Cycles et fluctuations économiques, Évaluation des actifs
Codes JEL : D3, D8, E2

NON-TECHNICAL SUMMARY

For several years now, wealth inequality has been attracting extensive discussions among policy-makers and researchers. There has been a broad consensus on the idea that households' resource allocation across assets is a key determinant of the wealth distribution. Earlier studies, for instance, have suggested that Knightian uncertainty (or, equivalently, model uncertainty or robustness) plays an important role in understanding household portfolio choice.

In this paper, we develop a tractable continuous-time general equilibrium model to analyze the implications of model uncertainty for the distribution of wealth. The economy consists of infinitely lived households differing in income and wealth. They face idiosyncratic income risk and can trade two assets, including a riskless bond and a risky real asset, to smooth consumption. Borrowing is permitted but subject to a constraint. In addition, households are confronted with model uncertainty related to the process driving the return of the risky asset, and they choose optimal policies that are robust to the model uncertainty.

We characterize the policy functions of both poor and rich households. The main findings are three-fold. First, the model distortion from uncertainty for both the poor and rich household is small, though for different reasons. Poor households face little distortion from uncertainty since they hold little of the risky asset due to the borrowing constraint. Rich households are, in contrast, little affected by uncertainty because their wealth is sufficient to insure against most, if not all, possible outcomes. The effect of model uncertainty reaches its maximum at some intermediate wealth level. Second, under certain conditions, robust policies accelerate the speed of convergence to the borrowing constraint. Lower wealth translates into a tighter borrowing constraint, which further constrains investment in the risky asset. These two forces together lead to a slower wealth accumulation. Third, the wealthier the households are, the smaller the reduction in the proportion of their wealth invested in the risky asset because of model uncertainty.

We further examine numerically the impacts of model uncertainty on wealth distribution and welfare. We find that robustness decreases the wealth share of the bottom 50 percent but increases that of the top 1 percent. Relaxing the borrowing constraint amplifies the distributional effects of model uncertainty. Finally, our welfare analysis shows that model uncertainty results in an aggregate welfare loss, unevenly distributed across households.

1. Introduction

Wealth inequality has attracted extensive discussions among policy-makers and researchers in recent years. There is a broad consensus that how households allocate resources across assets is a key determinant of wealth distribution. Earlier studies have suggested that Knightian uncertainty plays an important role in understanding household portfolio choice.[1] The objective of this paper is to analyze the implications of model uncertainty associated with risky investment for wealth distribution.

To do this, we develop a tractable continuous-time general equilibrium model. The model economy consists of infinitely lived households differing in their income and wealth. These households receive a periodic income that follows a two-state Poisson process, and they can trade two assets, including a riskless bond and a risky real asset, to smooth consumption. Borrowing is permitted but subject to a constraint. Households confront model uncertainty about the process driving the return of the risky asset, and they choose robust policies *à la* Hansen and Sargent (2008). Robustness results from a dynamic zero-sum game between a household and nature. The household makes a standard consumption-portfolio choice to maximize its lifetime utility. At the same time, nature chooses how severely to distort the risky return perceived by the household so as to minimize a distortion cost.

We characterize the policy functions of both poor and rich households. The main findings are three-fold. First, the size of model distortion chosen by nature varies non-monotonically with household wealth, reaching its maximum at some intermediate wealth level. Nature finds it optimal not to distort heavily the perceived risky return of poor households, as they hold few risky assets due to the borrowing constraint. For households in possession of large wealth, the benefit that comes from nature twisting their perception of the risky return is insignificant, so the distortion is also small. Second, we formulate the effects of model uncertainty on the speed at which the wealth of unlucky households hits the borrowing constraint. It is shown that under certain conditions, robustness accelerates the speed of convergence. Model uncertainty discourages these households from investing

[1] This paper uses "model uncertainty" and "robustness" interchangeably, both meaning "Knightian uncertainty".

in the risky asset due to their mistrust of the probability distribution underlying the risky return process. Lower wealth translates into a tighter borrowing constraint, which further constrains their investment in the risky asset. These two forces together lead to a slower wealth accumulation. Third, we derive the policy functions of the rich. For these households, the wealthier they are, the less nature distorts their perception, and thus the smaller is the reduction in the proportion of their wealth invested in the risky asset. In other words, robustness makes richer households even richer.

We prove the existence of a stationary equilibrium and examine numerically the impacts of model uncertainty on wealth distribution and welfare. There are three main findings. First, robustness decreases the wealth share of the bottom 50 percent but increases that of the top 1 percent. Our parameterized model generates a wealth distribution broadly in line with U.S. data. Second, a relaxation of the borrowing constraint amplifies the distributional effects of model uncertainty. The intuition is that in this case, since households can more easily borrow and invest in the risky asset, it is in the best interest of nature to pay the distortion cost and reduce the perceived excess return. Third, our welfare analysis shows that model uncertainty leads to a welfare loss for the rich but a gain for the poor. Overall, robustness induces a welfare loss equivalent to a 0.494 percent drop in aggregate wealth.[2]

Our paper is related to two strands of the literature. First, it is connected to the literature that examines the effects of model uncertainty on households' consumption-portfolio choice and the macroeconomy. Examples include Anderson, Hansen, and Sargent (2003), Maenhout (2004), Luo, Nie, and Young (2018), and Kasa and Lei (2018).[3] Recent work by Kasa and Lei (2018) is the study closest to ours. The current paper differs from theirs in the following two aspects. First, we bring attention to the interactions between model uncertainty and the borrowing constraint. To our best knowledge, this is the first paper that characterizes this interplay explicitly, which bears important macroeconomic implications. For example, a borrowing constraint changes the outcome of the dynamic

[2]This estimate is comparable to that in Ellison and Sargent (2015). Our study differs from theirs by quantifying the welfare effects of robustness from both aggregate and disaggregate perspectives.

[3]Empirical studies, e.g., Dimmock et al. (2016) and Brenner and Izhakian (2018), provide supporting evidence for the importance of Knightian uncertainty for understanding household portfolio choice.

zero-sum game, resulting in a non-monotonic relationship between model distortion and wealth. By contrast, the relationship is monotonic in Kasa and Lei (2018). Second, their paper answers the question of whether model uncertainty can explain the rise in top wealth shares in the U.S., while our main goal is to analyze the effects of robustness on the whole wealth distribution and evaluate its welfare implications for different households.

The current paper also contributes to the growing literature that analyzes the macroeconomic effects of household heterogeneity in continuous time.[4] Examples include Benhabib, Bisin, and Zhu (2016), Gabaix et al. (2016), Achdou et al. (2017), Cao and Luo (2017), Kaplan, Moll, and Violante (2018), Nuño and Moll (2018), and Toda and Walsh (2018). Our paper is most closely related to Achdou et al. (2017), upon which we build our model. The paper brings two contributions to this literature. First, we show that robustness provides a useful perspective for understanding both tails of wealth distribution. Second, we extend relevant results in Achdou et al. (2017) for the context of model uncertainty, and derive a novel formulation for its impact on the speed of convergence.

The rest of the paper is structured as follows. Section 2 describes the model. Section 3 characterizes the policy functions of both poor and rich households. Section 4 investigates the distributional and welfare impacts of model uncertainty. Section 5 concludes.

2. The Model

This section constructs a continuous-time general equilibrium economy populated by a continuum of heterogeneous households differing in their income and wealth. There are two assets in the economy: a riskless bond and a risky real asset.

2.1. Preferences.

Households have standard preferences over consumption as

$$U_0 = \mathbb{E}_0 \left[\int_0^{+\infty} e^{-\rho t} u(c_t) dt \right], \tag{2.1}$$

where ρ denotes the subjective discount factor and c_t consumption at date t. For analytical tractability, we specify the utility function $u(c)$ to be a constant relative risk aversion

[4] There is also a large literature that studies equilibrium models with heterogenous households in discrete time. Examples include Bewley (1986), Huggett (1993), Aiyagari (1994), Benhabib, Bisin, and Zhu (2015), Quadrini and Ríos-Rull (2015) and references therein.

(CRRA) utility

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma},\ \gamma > 0. \tag{2.2}$$

2.2. **Endowment.** Households receive an income z_t, which evolves stochastically over time. We assume that income follows a two-state Poisson process and takes values in $\{z_1, z_2\}$ with $z_2 > z_1$. The process switches from state 1 to state 2 with intensity λ_1 and from state 2 to state 1 with intensity λ_2.

2.3. **Market structure.** Households can trade two types of assets to smooth consumption. First, they have access to an instantaneously maturing riskless bond that pays an interest rate r_t. They can also trade a real risky asset with return R_t. This asset is real in the sense that each unit produces R_t units of physical output and only non-negative positions are allowed. The return of the risky asset varies over time according to

$$dR_t = Rdt + \sigma dW_t, \tag{2.3}$$

where R and σ are parameters. Denote b_t and k_t as a household's positions in the bond and the risky asset, respectively, and $a_t = b_t + k_t$ as its net worth.

Households can also borrow subject to a constraint

$$b_t \geq -\phi, \tag{2.4}$$

where $\phi \geq 0$. The borrowing constraint and the requirement of non-negative holdings of the risky asset can be summarized by the following condition:

$$0 \leq k_t \leq a_t + \phi, \tag{2.5}$$

where $a_t \geq \underline{a} = -\phi$ is the constraint imposed on net worth.

2.4. **Robust portfolio choice.** In this economy, households do not perfectly know the probability measure underlying the risky return process in (2.3). We capture this model uncertainty using the notion *à la* Hansen et al. (2006) and Hansen and Sargent (2008).[5]

[5]For the axiomatic foundations of model uncertainty, please refer to Gilboa and Schmeidler (1989), Maccheroni, Marinacci, and Rustichini (2006), and Strzalecki (2011).

When making their consumption-portfolio decisions, households consider multiple alternative probability measures and choose policies to obtain the highest expected utility under the worst case scenario. Robustness is achieved by assuming that each household plays a zero-sum game with nature. Nature distorts the drift term of the risky return process, while the household makes its portfolio choice taking the distorted return as given.

More precisely, let q^0 be a probability measure defined by the Brownian motion in the reference law of motion describing return process (2.3) and let q be an alternative law. The distance between the two laws is measured by the expected discounted log likelihood ratio, also called relative entropy

$$\mathcal{R}(q) = \rho \int_0^\infty e^{-\rho t} \left[\int \log \left(\frac{dq_t}{dq_t^0} \right) dq_t \right] dt = \frac{1}{2}\mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} h_t^2 dt \right], \tag{2.6}$$

where the second equality is due to the Girsanov Theorem and h_t is a square integrable and measurable process. The value of h_t represents the distortion chosen by nature. One can then view the alternative model q as induced by the following Brownian motion

$$d\tilde{W}_t = dW_t - h_t dt. \tag{2.7}$$

Consequently, the alternative risky return process is

$$dR_t = (R + \sigma h_t)\, dt + \sigma d\tilde{W}_t. \tag{2.8}$$

As a result, the dynamic budget constraints perceived by a household can be written as

$$da_t = (z_t + ra_t + k_t(R + \sigma h_t - r) - c_t)\, dt + \sigma k_t d\tilde{W}_t. \tag{2.9}$$

The objective of a household is to choose a consumption plan $\{c_t\}$ and an investment plan in the risky asset $\{k_t\}$ to maximize its lifetime utility, subject to budget constraint (2.9) and borrowing constraint (2.5). By contrast, nature chooses a distortion plan $\{h_t\}$ to minimize a distortion cost represented by the relative entropy $\mathcal{R}(q)$.[6] As such, the robust

[6]This form of cost functions has been used widely in the literature, e.g., Hansen and Sargent (2008).

consumption-portfolio choice problem can be formulated as

$$V_0 = \max_{\{c_t\},\{k_t\}} \min_{\{h_t\}} \mathbb{E}_0 \left[\int_0^{+\infty} e^{-\rho t} \left(u(c_t) + \frac{1}{2\varepsilon} {h_t}^2 \right) dt \right] \text{ s.t.} \tag{2.10}$$

$$da_t = (z_t + ra_t + k_t(R + \sigma h_t - r) - c_t)\, dt + \sigma k_t d\tilde{W}_t,$$

$$0 \leq k_t \leq a_t + \phi.$$

The parameter ε represents the robustness parameter, where $1/\varepsilon$ can be interpreted as a marginal cost of distorting the drift term of the risky return process. When $\varepsilon = 0$, the marginal cost is infinite and nature chooses $h = 0$ so that there is no doubt in the law of motion of the risky return. In this case, the model reduces to a standard Merton portfolio choice problem without robustness.

2.5. **Stationary equilibrium.** Denote j as one state of income and $-j$ as the other. A stationary equilibrium consists of value function $v_j(a)$, optimal consumption $c_j(a)$, optimal investment in the risky asset $k_j(a)$, optimal distortion $h_j\,(a)$, and density function $g_j(a)$ such that

(1) The value function $v_j(a)$ and policy functions $c_j(a)$, $k_j(a)$ and $h_j\,(a)$ together solve the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\rho v_j(a) = \max_{c, 0 \leq k \leq a+\phi} \min_h \left\{ \begin{array}{c} u(c) + \frac{1}{2\varepsilon} h^2 + v_j'(a)(z_j + ra + k(R + \sigma h - r) - c) \\ + \frac{1}{2} v_j''(a) \sigma^2 k^2 + \lambda_j (v_{-j}(a) - v_j(a)) \end{array} \right\}. \tag{2.11}$$

(2) The density function $g_j(a)$ satisfies the following Kolmogorov Forward (KF) equation:

$$0 = -\frac{\partial}{\partial a} \left[\hat{s}_j(a) g_j(a) \right] + \frac{1}{2} \frac{\partial^2}{\partial a^2} \left[\sigma^2 k_j(a)^2 g_j(a) \right] - \lambda_j g_j(a) + \lambda_{-j} g_{-j}(a), \tag{2.12}$$

where

$$\hat{s}_j(a) = z_j + ra + k_j(a)(R - r) - c_j(a). \tag{2.13}$$

(3) The bond market clears:

$$\sum_{j=1}^{2} \int_{\underline{a}}^{\infty} (a - k_j(a))\, g_j(a) da = 0. \tag{2.14}$$

3. Policy Functions

This section characterizes the impact of model uncertainty on the consumption-investment behavior of both wealth poor and rich households. The analysis provides the key insights on the implications of robustness for wealth distribution. The characterization takes advantage of a perturbation method related to the distortion parameter ε, which allows us to disentangle the effects of robustness in an analytical fashion.

3.1. **Portfolio choice of the poor.** We first consider the behavior of poor households, with the results summarized in the following proposition.

Proposition 3.1. *Suppose that $r < \rho$ at the steady state with $\underline{a} > -\frac{z_1}{r}$. Then the solution to the HJB equation (2.11) has the following properties:*

(1) *For $j = 1, 2$, $h_j(\underline{a}) = 0$, and there exists an $a_j^* > \underline{a}$ such that $h_j\left(a_j^*\right) \leq h_j(a)$ for $a \in [\underline{a}, \infty)$.*

(2) *Denote*

$$s_j(a) = z_j + ra + k_j(a)(R + \sigma h_j(a) - r) - c_j(a) \tag{3.15}$$

as the robust saving function. As $a \to \underline{a}$, $s_j(a)$ satisfies:

$$s_1(a) \sim -\sqrt{2\nu_1 (a - \underline{a})}, \tag{3.16}$$

where ν_1 is a constant defined by

$$\begin{aligned} \nu_1 \quad = \quad & \frac{(\rho - r)\, c_1(\underline{a})}{\gamma} + \lambda_1 (c_2(\underline{a}) - c_1(\underline{a})) \\ & - \frac{c_1(\underline{a})}{2\gamma} \left(\frac{R - r}{\sigma}\right)^2 \left(\frac{\gamma + 1}{\gamma} - \frac{1}{\gamma} \frac{c_1(\underline{a})\, c_1''(\underline{a})}{c_1'(\underline{a})^2}\right) \left(1 - 2\frac{\varepsilon}{\gamma} \frac{c_1(\underline{a})^{1-\gamma}}{c_1'(\underline{a})}\right). \end{aligned} \tag{3.17}$$

Part 1 of Proposition 3.1 shows the effects of wealth on the way nature distorts the return of the risky asset. The distortion is maximal at some intermediate wealth level. Nature does not distort the risky return perceived by constrained households because they do not hold any risky asset due to the borrowing constraint in (2.5) and thus have nothing to lose. Meanwhile, the impact of model distortion on rich households is inconsequential, so the distortion is also small.

Part 2 of Proposition 3.1 extends the results in Achdou et al. (2017) to the case with risky investment and model uncertainty. It characterizes the shape of the robust saving function of low-income households in the proximity of the borrowing constraint, where the saving policy behaves like $-\sqrt{\nu_1 a}$. As shown in equation (3.17), the value of ν_1 is determined by three different factors. The first and second terms on the right-hand side of the equation capture, respectively, the effects of intertemporal substitution and income uncertainty, whereas the third represents the impact from risky investment, taking robustness concerns into consideration.

In the model, households make their portfolio choice based on the perceived risky return process given in (2.8). However, their actual wealth accumulation is driven by the realized return and thus the realized saving function in (2.13), which we characterize in the following proposition.

Proposition 3.2. *Suppose that $r < \rho$ at the steady state with $\underline{a} > -\frac{z_1}{r}$. As $a \to \underline{a}$, the realized saving function $\hat{s}_j(a)$ satisfies:*

$$\hat{s}_1(a) \sim -\sqrt{2\hat{\nu}_1 (a - \underline{a})}, \tag{3.18}$$

where $\hat{\nu}_1$ is a constant given in the Appendix with

$$\hat{\nu}_1 < \nu_1. \tag{3.19}$$

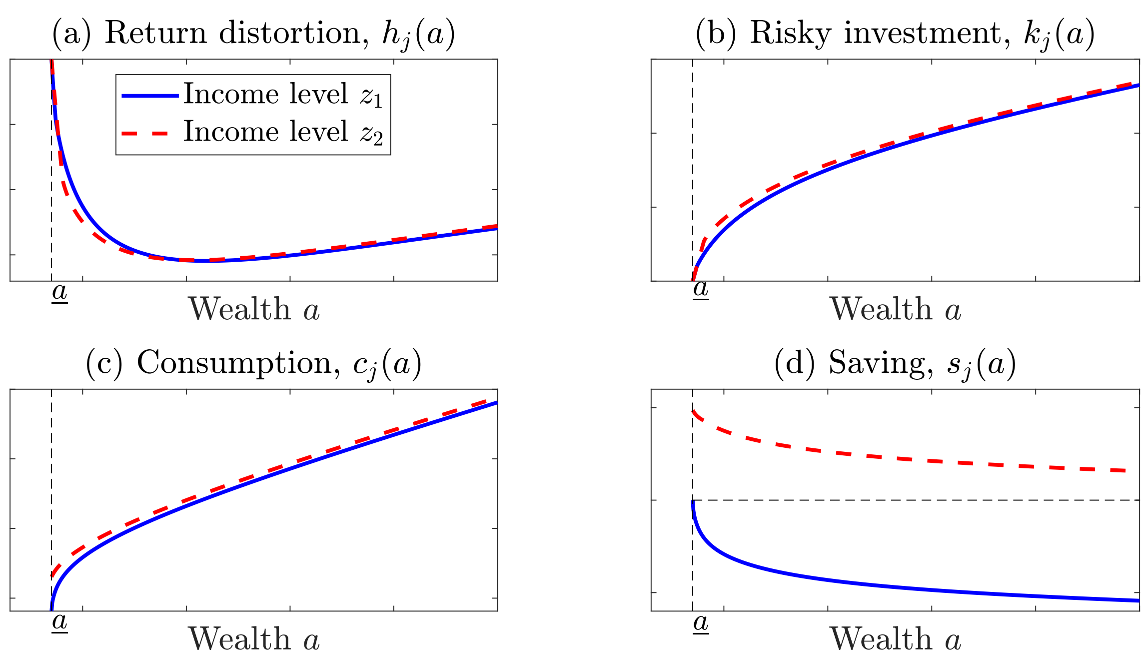
Using Proposition 3.2, we can show that the wealth of a household with initial wealth a_0 and successive low income draws z_1 hits the borrowing constraint in finite time T, where

$$T = \sqrt{\frac{2(a_0 - \underline{a})}{\hat{\nu}_1}}. \tag{3.20}$$

Therefore, $\hat{\nu}_1$ measures the speed of convergence to the borrowing constraint. The result is that $\hat{\nu}_1 < \nu_1$ is intuitive because the realized return from the risky asset is higher than what would be perceived by a household with robustness concerns. Thus, the speed of convergence is smaller than what it would be using the robust saving function.

We illustrate the individual policy functions in Figure 1. Panel (a) shows that the size of model distortion changes non-monotonically with wealth, which is consistent with the prediction in Proposition 3.1. By (2.8), the distortion alters the drift of the perceived risky return process from R to $R + \sigma h_j(a)$. The figure thus suggests that households in the two ends of the wealth distribution are less pessimistic than their middle counterparts. Panels (b) to (d) exhibit the behavior of risky investment, consumption and saving, respectively. As expected, the accumulation of wealth raises both investment and consumption but decreases saving. In addition, the investment and consumption functions of low-income households are concave near the borrowing constraint. This stems from the fact that as wealth approaches the constraint, it declines more rapidly than investment and consumption.

Figure 1: Policy functions



Model uncertainty affects the speed of convergence to the borrowing constraint. To characterize the impact explicitly, we compare our benchmark economy with an otherwise identical economy without robustness by calculating the difference between their respective convergence speeds. The results are presented in the following proposition.

Proposition 3.3. *Suppose that $r < \rho$ at the steady state with $\underline{a} > -\frac{z_1}{r}$. Denote*

$$c_j(a) = c_{j,0}(a) + \varepsilon c_{j,1}(a) + O(\varepsilon^2),\ j = 1, 2, \tag{3.21}$$

as the first-order approximation of the consumption function, where $c_{j,0}(a)$ represents the consumption function in the economy without robustness. It holds that

$$\hat{\nu}_1 \approx \nu_{1,0} + \varepsilon \nu_{1,1}, \tag{3.22}$$

where $\hat{\nu}_1$ is the speed of convergence in the benchmark economy,

$$\nu_{1,0} = \frac{(\rho - r)c_{1,0}(\underline{a})}{\gamma} + \lambda_1 (c_{2,0}(\underline{a}) - c_{1,0}(\underline{a})) - \frac{c_{1,0}(\underline{a})}{2\gamma} \left(\frac{R - r}{\sigma}\right)^2 \left(\frac{\gamma + 1}{\gamma} - \frac{1}{\gamma} \frac{c_{1,0}(\underline{a})\, c''_{1,0}(\underline{a})}{c'_{1,0}(\underline{a})^2}\right) \tag{3.23}$$

represents the speed of convergence in the economy without robustness, and $\nu_{1,1}$ is some constant defined in the Appendix.

The proof of Proposition 3.3 is based on a first-order Taylor expansion of the speed $\hat{\nu}_1$ around $\varepsilon = 0$, which corresponds to the case without model uncertainty.

Corollary 3.4. *Suppose that $r < \rho$ at the steady state with $\underline{a} > -\frac{z_1}{r}$. If the following three conditions are satisfied:*

(1) $c_{2,1}(\underline{a}) > c_{1,1}(\underline{a}) > 0$;

(2) $c'_{1,0}(a) > 0$ *and* $c''_{1,0}(a) < 0$;

(3) $\theta(\underline{a}) < 0$, *where $\theta(a)$ is a function defined in the Appendix,*

then it holds that

$$\hat{\nu}_1 > \nu_{1,0}. \tag{3.24}$$

The corollary says that the wealth of unlucky households in the benchmark economy hits the borrowing constraint at a faster rate than in the economy without model uncertainty. This result is driven by two forces. First, robustness induces households to believe that the excess return on the risky asset is lower than what it is in reality. Consequently, they invest less in the higher yielding asset, which decelerates wealth accumulation. Second, lower wealth translates into a tighter borrowing constraint. This further discourages the investment in the risky asset. The corollary immediately implies that, all else being equal, the presence of robustness concerns in portfolio choice increases the mass of poor households at any given point in time.

3.2. **Portfolio choice of the rich.** Next, we analyze the consumption and investment behavior of rich households. In this case, they will behave as in a problem without income and without a borrowing constraint. Before proceeding to the analysis of the policy functions, we first derive one auxiliary lemma concerning a homogeneity property of the value function defined in (2.11).

Lemma 3.5. *As $a \to \infty$, the value function solving the HJB equation (2.11) can be approximately written as*

$$v_j(a) \approx v_{j,0}(a) + \varepsilon v_{j,1}(a), \tag{3.25}$$

where the two functions $v_{j,0}(a)$ and $v_{j,1}(a)$ are such that for any $\xi > 0$,

$$v_{j,0}(\xi a) = \xi^{1-\gamma} v_{\xi,j,0}(a),\ v_{j,1}(\xi a) = \xi^{2(1-\gamma)} v_{\xi,j,1}(a), \tag{3.26}$$

with $v_{\xi,j,0}(a)$ and $v_{\xi,j,1}(a)$ satisfying two functional equations defined in the Appendix.

Using the lemma, the following proposition provides an analytical approximation for the policy functions when wealth is very large. The proof rests on the fact that borrowing constraint and income become irrelevant for the portfolio choice of rich households.

Proposition 3.6. *As $a \to \infty$, the individual policy functions solving the HJB equation (2.11) can be approximately written as*

$$c_j(a) \approx \alpha_0^{-\frac{1}{\gamma}} a - \varepsilon \frac{1}{\gamma} \alpha_0^{-\frac{1}{\gamma}-1} \alpha_1 a^{2-\gamma} \tag{3.27}$$

$$k_j(a) \approx \frac{R-r}{\gamma\sigma^2} a - \varepsilon \frac{R-r}{\sigma^2} \frac{\alpha_0^2 + \alpha_1(\gamma-1)}{\alpha_0 \gamma^2} a^{2-\gamma} \tag{3.28}$$

$$h_j(a) \approx -\varepsilon \frac{R-r}{\gamma\sigma^2} \alpha_0 a^{1-\gamma}, \tag{3.29}$$

where

$$\alpha_0 = \left(\frac{\rho-(1-\gamma)\,r}{\gamma}-\frac{1-\gamma}{2\gamma}\frac{(R-r)^2}{\gamma\sigma^2}\right)^{-\gamma} \tag{3.30}$$

$$\alpha_1 = \frac{\frac{1}{2}\alpha_0^2\left(\frac{R-r}{\gamma\sigma}\right)^2}{r-\alpha_0^{-\frac{1}{\gamma}}+\frac{1}{2}\left(\frac{R-r}{\gamma\sigma}\right)^2-\frac{\rho}{2(1-\gamma)}}. \tag{3.31}$$

Equation (3.29) shows that when the CRRA exceeds one, a widely accepted assumption in the literature, the size of model distortion declines with wealth and it diminishes gradually to zero as wealth approaches infinity. This is reminiscent of the observation in Panel (a) of Figure 1. Wealth discourages nature from distorting the risky return because the more wealth a household holds, the less pessimistic the household is about the return obtained from the risky investment.[7]

Dividing both sides of (3.28) by a yields

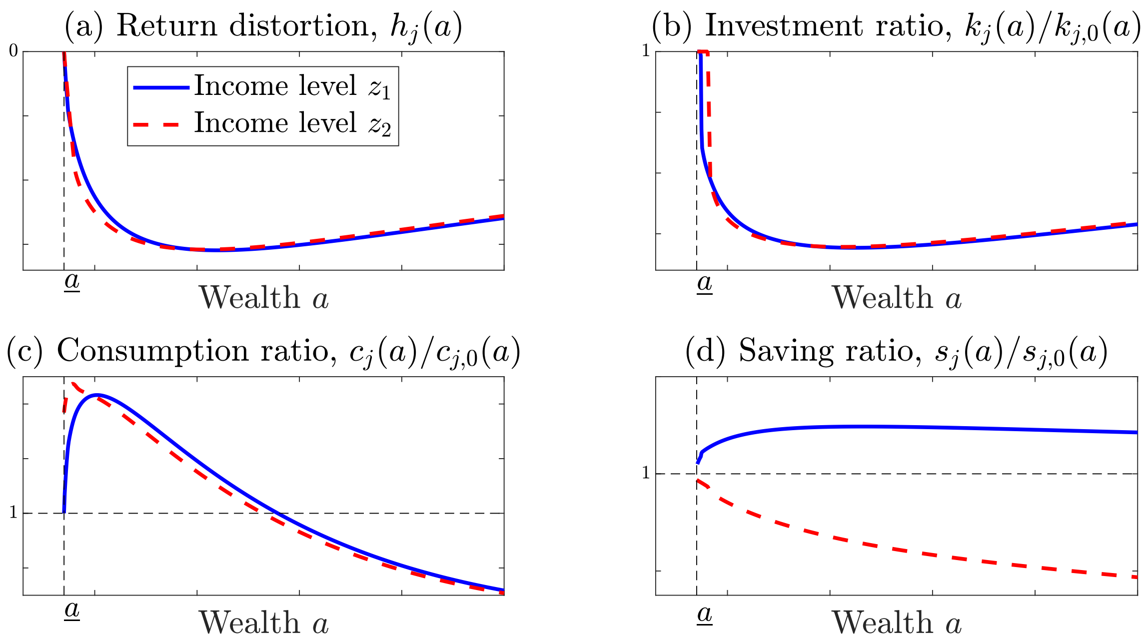
$$\frac{k_j\left(a\right)}{a}\approx\frac{R-r}{\gamma\sigma^2}-\varepsilon\frac{R-r}{\gamma\sigma^2}\frac{\alpha_0^2+\alpha_1(\gamma-1)}{\alpha_0\gamma}a^{1-\gamma}. \tag{3.32}$$

It shows that for the rich, the share of their wealth invested in the risky asset can be decomposed into two components. The first term corresponds to the standard Merton portfolio share, while the second captures the effects of model uncertainty. In the case with $\frac{\alpha_0^2+\alpha_1(\gamma-1)}{\alpha_0\gamma^2}>0$, robustness concerns cause rich investors to cut their investment in the risky asset, but the reduction is less pronounced for wealthier investors. In other words, richer households invest a larger share of their wealth in the higher yielding asset.

We conclude this section by comparing in Figure 2 the policy function $x_j\left(a\right)$ in the benchmark economy with its counterpart $x_{j,0}\left(a\right)$ in an otherwise identical economy except for model uncertainty. Panel (b) shows that robustness concerns reduce the risky investment across the board. The reduction, however, varies non-monotonically with wealth, and it follows closely the pattern of the distortion in Panel (a). Panel (c) displays the implications of model uncertainty for consumption. As discussed previously, robustness reduces the risky investment of households near the borrowing constraint, resulting in an

[7]Kasa and Lei (2018) derive similar robust policy functions in a Blanchard–Yaari framework over the entire state space. In our paper, these functions are only valid for the rich, and they take a different shape in the left tail due to the borrowing constraint.

Figure 2: The effects of robustness on policy functions



increase in their consumption. Contrarily, the consumption of the rich falls as a result of a reduction in net worth stemming from the decrease in their holdings of the higher yielding asset. Panel (d) depicts the reaction of saving to robustness. The ratio is above one for the low-income state and below one for the high-income state. As seen in Figure 1, saving is negative in the low-income state and positive in the high-income state, suggesting that the presence of model uncertainty slows wealth accumulation.

4. Wealth Distribution

This section studies the effects of model uncertainty for wealth distribution and welfare.

4.1. **Stationary distribution.** We first prove the existence of a stationary equilibrium by finding an interest rate that satisfies the bond market clearing condition (2.14).

Proposition 4.1. *There exists a stationary equilibrium in our benchmark economy.*

To illustrate the distributional implications of model uncertainty, we first parameterize the benchmark model according to U.S. data. We set the subjective discount rate ρ at

0.058 to match an average annual real interest rate of 4 percent. The risk aversion γ is fixed at 2, a value well within the consensus range of the parameter. Following Kasa and Lei (2018), we choose the expected return R of the risky asset equal to 0.059 and the standard deviation σ to 0.09. The parameters governing the income process are taken from Achdou et al. (2017), where the two income states z_1 and z_2 are set at 0.4 and 0.6, respectively, and the transition intensity between the states $\lambda_1 = \lambda_2 = 0.5$. The borrowing constraint parameter ϕ equals 1.2, a value broadly in line with Huggett (1993) and Achdou et al. (2017). The robustness parameter ε is calibrated based on Anderson et al. (2003) and Kasa and Lei (2018), and it is set at 0.067. The corresponding state dependent detection error probabilities are all above 45 percent, suggesting the empirical plausibility of model uncertainty. We compute the stationary equilibrium numerically using a finite difference method developed in Achdou et al. (2017).

Table 1 compares the stationary wealth distribution in the benchmark economy with that in an otherwise identical economy without robustness. Pertaining to the left tail, robustness increases the fraction of households with negative wealth and decreases the wealth share of the bottom 50 percent. The results are in line with the findings in Corollary 3.4, which shows that robustness increases the speed of convergence to the borrowing constraint. Meanwhile, robustness generates a larger wealth share of the top 1 percent because richer households invest an unevenly larger portion of their wealth in the risky asset. Putting this together, model uncertainty leads to a higher wealth concentration, which is also reflected in the rise of the wealth Gini coefficient. Empirically, introducing robustness helps improve the model's fit of the U.S. wealth distribution, particularly the two tails.

The distributional effects of robustness are further illustrated in Figure 3. Panel (a) shows that the cumulative distribution function of wealth in the robust economy is first-order stochastically dominated by its counterpart in the nonrobust economy. This is because robustness reduces net worth of all households. Panel (b) demonstrates that model uncertainty shifts the Lorenz curve outward, indicating a rise in wealth inequality.

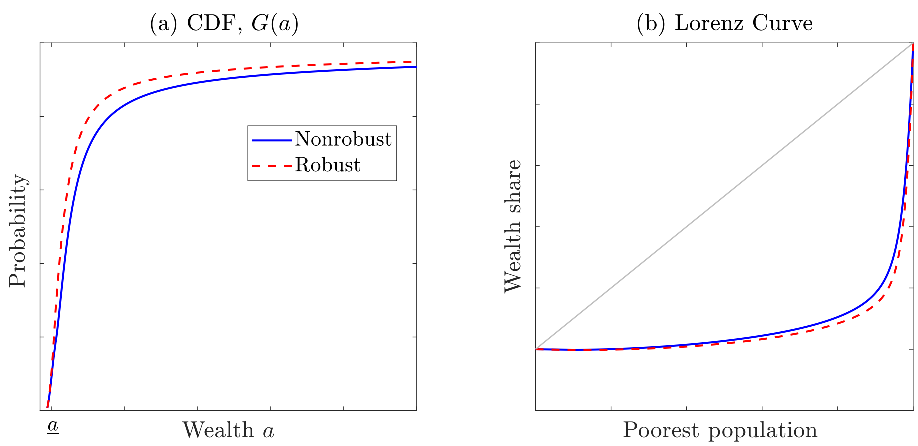
4.2. **Comparative statics.** In what follows, we conduct a sensitivity analysis of how the computed distributional effects of model uncertainty depend on parameters. In the

Table 1: Stationary distribution

Object	Data	Nonrobust	Robust	Change (%)
$\mathbb{P}(a < 0)$	0.100	0.093	0.109	16.7
$[0, 50)$	0.018	0.027	0.019	-31.5
$[50, 90)$	0.251	0.156	0.128	-18.1
$[90, 99)$	0.382	0.563	0.544	-3.5
$[99, 100]$	0.350	0.253	0.309	22.2
Gini	0.860	0.852	0.880	3.3

Notes : The source of the data is the Survey of Consumer Finances. Robust and Nonrobust correspond to the economy with and without robustness, respectively. The last column reports the percent change of each statistics between the two economies.

Figure 3: Distributional statistics



analysis, except for parameters explicitly under investigation, the rest are all set at their baseline values.

The results are presented in Table 2. First, a rise in ϕ, which means a relaxed borrowing constraint, makes the distributional impact of model uncertainty more pronounced. The intuition is that in this case, since households can more easily borrow and invest in the risky asset, it is in the best interest of nature to pay the distortion cost and reduce the perceived excess return. Second, a higher expected return of the risky asset strengthens the distributional effects of robustness relative to the baseline scenario. All else equal, households are more willing to invest in the risky asset, in which case it is optimal for nature to create a bigger downward bias in the perceived return. Third, we consider a change in the volatility of the risky return. As expected, a decrease in σ generates similar outcomes as an increase in R. Summing up, our comparative statics exercise shows that robustness increases wealth inequality under plausible model parameterizations.

Table 2: Comparative statics

Object	$\phi = 2$			$R = 0.06$			$\sigma = 0.08$		
	Nonrobust	Robust	Change	Nonrobust	Robust	Change	Nonrobust	Robust	Change
$\mathbb{P}(a < 0)$	0.196	0.266	35.5	0.083	0.100	20.4	0.079	0.085	7.8
$[0, 50)$	0.013	0.001	-92.8	0.032	0.018	-43.3	0.032	0.020	-38.9
$[50, 90)$	0.153	0.124	-19.1	0.175	0.128	-26.8	0.187	0.138	-26.1
$[90, 99)$	0.587	0.581	-1.0	0.568	0.581	2.4	0.579	0.607	4.9
$[99, 100]$	0.248	0.293	18.5	0.227	0.272	20.1	0.203	0.235	15.5
Gini	0.874	0.907	3.8	0.834	0.877	5.1	0.828	0.868	4.8

Notes : Robust and Nonrobust correspond to the economy with and without robustness, respectively. The last column of each panel reports the percent change of each statistics between the two economies.

4.3. **Welfare cost of model uncertainty.** The previous analysis suggests that portfolio responses to model uncertainty vary considerably across households, suggesting a possible heterogeneity in the way they evaluate robustness. In what follows, we examine its welfare implications. To do this, we compute the fraction of wealth, $\delta_j(a)$, a household with wealth a is willing to give up in the nonrobust economy to be as well off as in an otherwise identical economy with robustness. As such, $\delta_j(a)$ satisfies

$$v_{0,j}((1 - \delta_j(a))a) = v_j(a),\ j = 1, 2, \tag{4.33}$$

where $v_{0,j}(a)$ and $v_j(a)$ represent the value functions in the nonrobust and robust economies, respectively. A positive value of $\delta_j(a)$ implies that the household is worse off with the existence of model uncertainty, and vice versa.

Table 3: Welfare cost of model uncertainty

Object	Cost (%)
$[0, 50)$	-2.150
$[50, 90)$	1.140
$[90, 99)$	0.562
$[99, 100]$	0.228
Overall	0.494

Table 3 summarizes the disaggregate and aggregate welfare effects of model uncertainty. We find that robustness implies a welfare loss for the rich but a gain for the poor. This result can be understood in light of Figure 2, which shows that robustness lowers the consumption of the former but raises that of the latter. Overall, model uncertainty induces a welfare loss equivalent to a 0.494 percent drop in aggregate wealth.

5. Conclusion

This paper examines the implications of model uncertainty for wealth distribution in a tractable continuous-time general equilibrium model. We find that the size of the model distortion chosen by nature varies non-monotonically with household wealth. Robustness generates a larger concentration of wealth due to two factors. It increases the speed at which the wealth of unlucky households hits the borrowing constraint. It also leads richer households to invest a disproportionally larger share of wealth in the higher yielding asset. Robustness implies a gain for the poor but a loss for the rich.

Our study shows that model uncertainty is an important source of changes in the cross-sectional distribution of key macroeconomic variables, due to the differential responses from households located at different parts of the wealth distribution. For policy-makers, it might thus be important to take into account households' robustness concerns when designing and conducting their policies. This is especially the case for those policies bearing uneven effects on household wealth since model uncertainty could greatly amplify their consequences, as suggested in the paper.

To illustrate the distributional effects of robustness, the model is deliberately kept simple. The mechanism proposed in the paper, however, opens the door to a proper quantitative analysis. For example, an emerging body of evidence highlights that in order to understand the aggregate economic activities during the Great Recession, it is crucial for a model to capture the large fraction of poor households.[8] Model uncertainty provides a useful channel through which a substantial share of the population become wealth poor.

[8]Existing proposals include heterogeneous preferences and rich earning processes, e.g., Krueger, Mitman, and Perri (2016) and De Nardi and Fella (2017).

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Appendix A. Proof of Proposition 3.1

Proof. To prove the proposition, we first show that $s_1(a) = 0$, and there exists an $\delta > 0$ such that $s_1(a) < 0$ for all $a \in (\underline{a}, \underline{a} + \delta)$. The proof of $s_1(\underline{a}) = 0$ is straightforward and is skipped here. We next prove $s_1(\underline{a}) < 0$ in a neighborhood of the borrowing constraint. The FOCs of the HJB equation (2.11) associated with c, k, h are, respectively,

$$u'(c_j(a)) = v_j'(a) \tag{A.34}$$

$$v_j'(a)(R + \sigma h_j(a) - r) + v_j''(a)\sigma^2 k_j(a) = 0 \tag{A.35}$$

$$h_j(a) + \varepsilon\sigma v_j'(a) k_j(a) = 0. \tag{A.36}$$

Note since $0 \leq k_1(a) \leq a - \underline{a}$, it follows that $\lim_{a \to \underline{a}} k_1(a) = 0$, and by (A.36) $\lim_{a \to \underline{a}} h_1(a) = 0$, where the last equality uses the fact that $v_1'(\underline{a}) = \lim_{a \to \underline{a}} u'(c_1(a)) < \infty$. Indeed,

$$\lim_{a \to \underline{a}} c_1(a) = \lim_{a \to \underline{a}} z_1 + ra + k_1(a)(R + \sigma h_1(a) - r) - s_1(a) = z_1 + r\underline{a},$$

which is positive by assumption. Combining (2.5), (A.35) and (A.36) yields

$$k_j(a) = \min\left\{\frac{R - r}{\sigma^2}\frac{v_j'(a)}{\varepsilon v_j'^2 - v_j''(a)}, a + \phi\right\}. \tag{A.37}$$

As a consequence, we have

$$0 = \lim_{a \to \underline{a}} k_1(a) = \lim_{a \to \underline{a}} -\frac{v_1'(a)}{v_1''(a)}\frac{R + \sigma h_1(a) - r}{\sigma^2} = -\frac{u'(c_1(\underline{a}))}{u''(c_1(\underline{a}))c_1'(\underline{a})}\frac{R - r}{\sigma^2},$$

meaning that $c_1'(\underline{a}) = \infty$. Because $0 \leq k_1(a) \leq a - \underline{a}$ and $k_1(\underline{a}) = 0$, it holds that

$$k_1'(\underline{a}) = \lim_{a \to \underline{a}} \frac{k_1(a) - k_1(\underline{a})}{a - \underline{a}} = \lim_{a \to \underline{a}} \frac{k_1(a)}{a - \underline{a}} \leq \lim_{a \to \underline{a}} \frac{a - \underline{a}}{a - \underline{a}} = 1,$$

i.e., $k_1'(\underline{a})$ is bounded. Differentiating (A.36) with respect to a yields

$$h_j'(a) = -\varepsilon\sigma\left(v_j''(a)k_j(a) + v_j'(a)k_j'(a)\right). \tag{A.38}$$

It then follows that

$$\begin{aligned}
\lim_{a\to\underline{a}} s_1'(a) &= \lim_{a\to\underline{a}} r + k_1'(a)(R+\sigma h_1(a)-r) + k_1(a)\sigma h_1'(a) - c_1'(a) \qquad (A.39)\\
&= \lim_{a\to\underline{a}} r + k_1'(a)(R+\sigma h_1(a)-r) - \varepsilon\sigma^2 k_1(a)\begin{pmatrix} v_1''(a)k_1(a)+ \\ v_1'(a)k_1'(a)\end{pmatrix} - c_1'(a)\\
&= -\infty,
\end{aligned}$$

where the last equality stems from the fact that $\lim_{a\to\underline{a}} k_1(a) = \lim_{a\to\underline{a}} h_1(a) = 0$, and the two limits $\lim_{a\to\underline{a}} k_1'(a)$ and $\lim_{a\to\underline{a}} v_1''(a)k_1(a) = -\lim_{a\to\underline{a}} v_1'(a)\frac{R+\sigma h_1(a)-r}{\sigma^2}$ are bounded. Since $s_1(\underline{a}) = 0$, it holds that there exists $\delta > 0$ such that $s_1(a) < 0$ for $a \in (\underline{a}, \underline{a}+\delta)$.

With the aid of the above proof, the existence of a global minimizer of $h_j(a)$ follows directly from the solution to the equation $h_j'(a) = 0$, where $h_j'(a)$ is given in (A.38).

We now proceed to derive the approximate analytical solution of $s_1(a)$ near the borrowing constraint. By (A.39), we have

$$\begin{aligned}
\lim_{a\to\underline{a}}\left(s_1'(a)+c_1'(a)\right)s_1(a) &= \lim_{a\to\underline{a}}\left(r + k_1'(a)(R+\sigma h_1(a)-r) + k_1(a)\sigma h_1'(a)\right)s_1(a)\\
&= \lim_{a\to\underline{a}}\begin{pmatrix} r + k_1'(a)(R+\sigma h_1(a)-r) \\ -\varepsilon\sigma^2 k_1(a)\left(v_1''(a)k_1(a)+v_1'(a)k_1'(a)\right)\end{pmatrix} s_1(a) = 0,
\end{aligned}$$

which implies that

$$\lim_{a\to\underline{a}} s_1'(a)s_1(a) = -\lim_{a\to\underline{a}} c_1'(a)s_1(a). \qquad (A.40)$$

We next derive the limit on the right-hand side. The Euler equation of problem (2.11) is

$$\begin{aligned}
\rho - r &= \frac{v_1''(a)}{v_1'(a)}s_1(a) + \frac{1}{2}\frac{v_1'''(a)}{v_1'(a)}\sigma^2 k_1^2(a) + \lambda_1\left(\frac{v_2'(a)}{v_1'(a)}-1\right)\\
&= \frac{v_1''(a)}{v_1'(a)}s_1(a) + \frac{1}{2}\frac{v_1'''(a)}{v_1'(a)}\sigma^2\left(\frac{R-r}{\sigma^2}\frac{v_1'(a)}{\varepsilon v_1'(a)^2 - v_1''(a)}\right)^2 + \lambda_1\left(\frac{v_2'(a)}{v_1'(a)}-1\right)\\
&\approx \frac{v_1''(a)}{v_1'(a)}s_1(a) + \frac{1}{2}\frac{v_1'''(a)}{v_1'(a)}\left(\frac{R-r}{\sigma}\right)^2\left(-\frac{v_1'(a)}{v_1''(a)} - \varepsilon\frac{v_1'(a)^3}{v_1''(a)^2}\right)^2 + \lambda_1\left(\frac{v_2'(a)}{v_1'(a)}-1\right)\\
&\approx \frac{v_1''(a)s_1(a)}{v_1'(a)} + \frac{(R-r)^2}{2\sigma^2}\frac{v_1'''(a)v_1'(a)}{v_1''(a)^2}\left(1+2\varepsilon\frac{v_1'(a)^2}{v_1''(a)}\right) + \lambda_1\left(\frac{v_2'(a)}{v_1'(a)}-1\right) \qquad (A.41)
\end{aligned}$$

By (A.34) and the functional form of u, we have

$$v_1'(a) = u'(c_1(a)) = c_1(a)^{-\gamma}$$
$$v_1''(a) = u''(c_1(a))\, c_1'(a) = -\gamma c_1(a)^{-\gamma-1} c_1'(a) \tag{A.42}$$
$$v_1'''(a) = u'''(c_1(a))\, c_1'(a)^2 + u''(c_1(a))\, c_1''(a) = \gamma(\gamma+1)\, c_1(a)^{-\gamma-2} c_1'(a)^2 - \gamma c_1(a)^{-\gamma-1} c_1''(a).$$

Substituting them into (A.41) and rearranging lead to

$$\begin{aligned} \rho - r \;\; = \;\; & -\gamma c_1(a)^{-1} c_1'(a)\, s_1(a) - \lambda_1 \gamma c_1(a)^{-1} (c_2(a) - c_1(a)) \\ & + \frac{1}{2}\left(\frac{R-r}{\sigma}\right)^2 \left(\frac{\gamma+1}{\gamma} - \frac{1}{\gamma} c_1(a) \frac{c_1''(a)}{c_1'(a)^2}\right)\left(1 - 2\varepsilon \frac{1}{\gamma} \frac{c_1(a)^{1-\gamma}}{c_1'(a)}\right). \end{aligned} \tag{A.43}$$

This together with (A.39) imply that

$$\lim_{a \to \underline{a}} s_1'(a)\, s_1(a) = \nu_1,$$

where ν_1 is defined in (3.17). As a consequence,

$$s_1(a)^2 \approx s_1(\underline{a})^2 + 2 s_1(\underline{a})\, s_1'(\underline{a})\,(a - \underline{a}) = 2\nu_1 (a - \underline{a}),$$

implying (3.16). □

Appendix B. Proof of Proposition 3.2

Proof. The derivation of $\hat{\nu}_1$ is similar to that of ν_1. First, it is straightforward to show that

$$\lim_{a \to \underline{a}} \hat{s}_1'(a)\, \hat{s}_1(a) = -\lim_{a \to \underline{a}} c_1'(a)\, \hat{s}_1(a). \tag{B.44}$$

Next, we find the limit on the right-hand side of the above equation. Equations (2.13) and (3.15) imply that

$$s_1(a) = \hat{s}_1(a) + \sigma k_j(a)\, h_j(a).$$

Plugging it into the Euler equation of problem (2.11) and rearranging yields

$$\begin{aligned}\rho - r &= \frac{v_1''(a)}{v_1'(a)}\left(\hat{s}_1(a) + \sigma k_j(a)\, h_j(a)\right) + \frac{1}{2}\frac{v_1'''(a)}{v_1'(a)}\sigma^2 k_1^2(a) + \lambda_1\left(\frac{v_2'(a)}{v_1'(a)} - 1\right) \\ &\approx \frac{v_1''(a)\,\hat{s}_1(a)}{v_1'(a)} + \frac{(R-r)^2}{2\sigma^2}\frac{v_1'''(a)\, v_1'(a)}{v_1''(a)^2}\left(1 + 2\varepsilon\left(\begin{array}{c}\frac{v_1'(a)^2}{v_1''(a)} \\ -\frac{v_1'(a)v_1''(a)}{v_1'''(a)}\end{array}\right)\right) \qquad \text{(B.45)} \\ &\quad + \lambda_1\left(\frac{v_2'(a)}{v_1'(a)} - 1\right).\end{aligned}$$

Using expressions in (A.42) in (B.45) results in

$$\begin{aligned}\rho - r &= -\gamma c_1(a)^{-1} c_1'(a)\,\hat{s}_1(a) - \lambda_1\gamma c_1(a)^{-1}\left(c_2(a) - c_1(a)\right) \qquad \text{(B.46)} \\ &\quad + \frac{1}{2}\left(\frac{R-r}{\sigma}\right)^2\left(\frac{\gamma+1}{\gamma} - \frac{1}{\gamma}c_1(a)\frac{c_1''(a)}{c_1'(a)^2}\right)\left(1 + 2\varepsilon\left(\begin{array}{c}-\frac{1}{\gamma}\frac{c_1(a)^{1-\gamma}}{c_1'(a)} \\ +\frac{c_1(a)^{1-\gamma}c_1'(a)}{(\gamma+1)c_1'(a)^2 - c_1''(a)c_1(a)}\end{array}\right)\right).\end{aligned}$$

Equation (B.46) together with (B.44) imply that

$$\lim_{a \to \underline{a}} \hat{s}_1'(a)\,\hat{s}_1(a) = \hat{\nu}_1,$$

where

$$\hat{\nu}_1 = \nu_1 - \varsigma$$

with ν_1 given in (3.17) and

$$\varsigma = \frac{c_1(\underline{a})}{2\gamma}\left(\frac{R-r}{\sigma}\right)^2\left(\frac{\gamma+1}{\gamma} - \frac{1}{\gamma}\frac{c_1(\underline{a})\,c_1''(\underline{a})}{c_1'(\underline{a})^2}\right)\left(2\varepsilon\frac{c_1(\underline{a})^{1-\gamma}c_1'(\underline{a})}{(\gamma+1)c_1'(\underline{a})^2 - c_1''(\underline{a})\,c_1(\underline{a})}\right) > 0.$$

Thus, we have $\hat{\nu}_1 < \nu_1$, and $\hat{s}_1(a)^2 \approx \hat{s}_1(\underline{a})^2 + 2\hat{s}_1(\underline{a})\,\hat{s}_1'(\underline{a})(a - \underline{a}) = 2\hat{\nu}_1(a - \underline{a})$. □

Appendix C. Proof of Proposition 3.3

Proof. First, we rewrite (B.46) as

$$\begin{aligned}-c_1'(a)\,\hat{s}_1(a) &= \frac{c_1(a)(\rho - r)}{\gamma} + \lambda_1\left(c_2(a) - c_1(a)\right) \qquad \text{(C.47)} \\ &\quad - \frac{c_1(a)}{2\gamma}\left(\frac{R-r}{\sigma}\right)^2\left(\frac{\gamma+1}{\gamma} - \frac{1}{\gamma}c_1(a)\frac{c_1''(a)}{c_1'(a)^2}\right)\left(1 + 2\varepsilon\left(\begin{array}{c}-\frac{1}{\gamma}\frac{c_1(a)^{1-\gamma}}{c_1'(a)} \\ +\frac{c_1(a)^{1-\gamma}c_1'(a)}{(\gamma+1)c_1'(a)^2 - c_1''(a)c_1(a)}\end{array}\right)\right).\end{aligned}$$

Denote the above equation as

$$\eta(a) = -c_1'(a)\, s_1(a) = \eta_1(a) + \eta_2(a) + \eta_3(a), \tag{C.48}$$

where

$$\eta_1(a) = \frac{c_1(a)(\rho - r)}{\gamma} \tag{C.49}$$

$$\eta_2(a) = \lambda_1 \left(c_2(a) - c_1(a)\right) \tag{C.50}$$

$$\eta_3(a) = -\frac{c_1(a)}{2\gamma}\left(\frac{R-r}{\sigma}\right)^2 \left(\begin{array}{c} \frac{\gamma+1}{\gamma} - \\ \frac{1}{\gamma} c_1(a) \frac{c_1''(a)}{c_1'(a)^2} \end{array}\right) \left(1 + 2\varepsilon \left(\begin{array}{c} -\frac{1}{\gamma}\frac{c_1(a)^{1-\gamma}}{c_1'(a)} \\ + \frac{c_1(a)^{1-\gamma} c_1'(a)}{(\gamma+1)c_1'(a)^2 - c_1''(a) c_1(a)} \end{array}\right)\right) \tag{C.51}$$

Note (3.17) implies $\eta(\underline{a}) = \hat{\nu}_1$. Next, we find in order the first-order approximation of functions $\eta_i(a)$, $i = 1, 2, 3$, around $\varepsilon = 0$. By (C.49) and (3.21), we have

$$\eta_1(a) = \frac{c_1(a)(\rho - r)}{\gamma} \approx \frac{(c_{1,0}(a) + \varepsilon c_{1,1}(a))(\rho - r)}{\gamma} = \eta_{1,0}(a) + \varepsilon \eta_{1,1}(a), \tag{C.52}$$

where

$$\eta_{1,0}(a) = \frac{c_{1,0}(a)(\rho - r)}{\gamma}, \eta_{1,1}(a) = \frac{c_{1,1}(a)(\rho - r)}{\gamma}. \tag{C.53}$$

By the same token, combining (C.50) and (3.21) yields

$$\begin{aligned} \eta_2(a) &= \lambda_1\left(c_2(a) - c_1(a)\right) \approx \lambda_1(c_{2,0}(a) + \varepsilon c_{2,1}(a) - c_{1,0}(a) - \varepsilon c_{1,1}(a)) \\ &= \lambda_1(c_{2,0}(a) - c_{1,0}(a)) + \varepsilon\lambda_1(c_{2,1}(a) - c_{1,1}(a)) = \eta_{2,0}(a) + \varepsilon\eta_{2,1}(a), \end{aligned} \tag{C.54}$$

where

$$\eta_{2,0}(a) = \lambda_1(c_{2,0}(a) - c_{1,0}(a)), \eta_{2,1}(a) = \lambda_1(c_{2,1}(a) - c_{1,1}(a)). \tag{C.55}$$

Finally, we can approximate $\eta_3(a)$ in (C.51) as

$$\begin{aligned}
\eta_3(a) &= -\frac{c_1(a)}{2\gamma}\left(\frac{R-r}{\sigma}\right)^2\left(\frac{\gamma+1}{\gamma}-\frac{1}{\gamma}c_1(a)\frac{c_1''(a)}{c_1'(a)^2}\right)\left(1+2\varepsilon\left(\begin{array}{c} -\frac{1}{\gamma}\frac{c_1(a)^{1-\gamma}}{c_1'(a)} \\ +\frac{c_1(a)^{1-\gamma}c_1'(a)}{(\gamma+1)c_1'(a)^2-c_1''(a)c_1(a)}\end{array}\right)\right) \\
&= -\frac{1}{2\gamma}\left(\frac{R-r}{\sigma}\right)^2\left(\frac{\gamma+1}{\gamma}c_1(a)-\frac{1}{\gamma}c_1(a)^2\frac{c_1''(a)}{c_1'(a)^2}\right)\left(1+2\varepsilon\left(\begin{array}{c} -\frac{1}{\gamma}\frac{c_1(a)^{1-\gamma}}{c_1'(a)} \\ +\frac{c_1(a)^{1-\gamma}c_1'(a)}{(\gamma+1)c_1'(a)^2-c_1''(a)c_1(a)}\end{array}\right)\right) \\
&\approx -\frac{1}{2\gamma}\left(\frac{R-r}{\sigma}\right)^2\left(\begin{array}{c}\frac{(\gamma+1)(c_{1,0}(a)+\varepsilon c_{1,1}(a))}{\gamma} \\ -\frac{1}{\gamma}(c_{1,0}(a)+\varepsilon c_{1,1}(a))^2\frac{c_{1,0}''(a)+\varepsilon c_{1,1}''(a)}{(c_{1,0}'(a)+\varepsilon c_{1,1}'(a))^2}\end{array}\right) \\
&\quad\times\left(1+2\varepsilon\left(\begin{array}{c} -\frac{1}{\gamma}\frac{(c_{1,0}(a)+\varepsilon c_{1,1}(a))^{1-\gamma}}{c_{1,0}'(a)+\varepsilon c_{1,1}'(a)} \\ +\frac{(c_{1,0}(a)+\varepsilon c_{1,1}(a))^{1-\gamma}\left(c_{1,0}'(a)+\varepsilon c_{1,1}'(a)\right)}{(\gamma+1)(c_{1,0}'(a)+\varepsilon c_{1,1}'(a))^2-(c_{1,0}''(a)+\varepsilon c_{1,1}''(a))(c_{1,0}'(a)+\varepsilon c_{1,1}'(a))}\end{array}\right)\right) \\
&\approx \eta_{3,0}(a)+\varepsilon\eta_{3,1}(a), \qquad \text{(C.56)}
\end{aligned}$$

where

$$\eta_{3,0}(a) = -\frac{1}{2\gamma}\left(\frac{R-r}{\sigma}\right)^2\left(\frac{\gamma+1}{\gamma}c_{1,0}(a)-\frac{1}{\gamma}c_{1,0}(a)^2\frac{c_{1,0}''(a)}{c_{1,0}'(a)^2}\right), \eta_{3,1}(a) = \eta_{3,0}(a)\tau(a). \qquad \text{(C.57)}$$

Here the function $\tau(a)$ is given by

$$\begin{aligned}
\tau(a) &= \frac{\theta(a)}{(\gamma+1)c_{1,0}(a)c_{1,0}'(a)^3-c_{1,0}(a)^2c_{1,0}'(a)c_{1,0}''(a)} \\
&\quad -\frac{2}{\gamma}\frac{c_{1,0}(a)^{1-\gamma}}{c_{1,0}'(a)}+\frac{2c_{1,0}(a)^{1-\gamma}c_{1,0}'(a)}{(\gamma+1)c_{1,0}'(a)^2-c_{1,0}(a)c_{1,0}''(a)}, \\
\theta(a) &= (\gamma+1)c_{1,1}(a)c_{1,0}'(a)^3-\left(\begin{array}{c}2c_{1,0}(a)c_{1,1}(a)c_{1,0}''(a) \\ +c_{1,0}(a)^2c_{1,1}''(a)\end{array}\right)c_{1,0}'(a)+2c_{1,0}(a)^2c_{1,0}''(a)c_{1,1}'(a).
\end{aligned}$$

Substituting (C.52), (C.54) and (C.56) into (C.48) and taking the limit of a to $\underline{a}$ lead to

$$\begin{aligned}
\hat{\nu}_1 &= \lim_{a\to\underline{a}}\eta(a) = \lim_{a\to\underline{a}}\sum_{i=1}^{3}\eta_i(a) \approx \lim_{a\to\underline{a}}\sum_{i=1}^{3}\left(\eta_{i,0}(a)+\varepsilon\eta_{i,1}(a)\right) \\
&= \sum_{i=1}^{3}\eta_{i,0}(\underline{a})+\varepsilon\sum_{i=1}^{3}\eta_{i,1}(\underline{a}). \qquad \text{(C.58)}
\end{aligned}$$

Note that

$$\sum_{i=1}^{3} \eta_{i,0}\left(\underline{a}\right) = \left(\begin{array}{c} \frac{c_{1,0}(\underline{a})(\rho-r)}{\gamma} + \lambda_1(c_{2,0}(\underline{a}) - c_{1,0}(\underline{a})) \\ -\frac{1}{2\gamma}\left(\frac{R-r}{\sigma}\right)^2 \left(\frac{(\gamma+1)c_{1,0}(\underline{a})}{\gamma} - \frac{1}{\gamma}c_{1,0}(\underline{a})^2 \frac{c''_{1,0}(\underline{a})}{c'_{1,0}(\underline{a})^2}\right) \end{array} \right) = v_{1,0}. \tag{C.59}$$

Thus, combining (C.58) and (C.59) yields (3.22) as desired, where

$$v_{1,1} = \sum_{i=1}^{3} \eta_{i,1}\left(\underline{a}\right). \tag{C.60}$$

□

Appendix D. Proof of Corollary 3.4

Proof. By (3.22), it suffices to prove $v_{1,1} > 0$. Because $c_{1,1}(\underline{a}) > 0$, we have

$$\eta_{1,1}\left(\underline{a}\right) = \frac{c_{1,1}(\underline{a})(\rho - r)}{\gamma} > 0.$$

Meanwhile, the condition $c_{2,1}\left(\underline{a}\right) > c_{1,1}\left(\underline{a}\right)$ implies that

$$\eta_{2,1}(\underline{a}) = \lambda_1(c_{2,1}(\underline{a}) - c_{1,1}(\underline{a})) > 0.$$

Since $c''_{1,0}(a) < 0$, $\eta_{3,0}\left(\underline{a}\right) < 0$ by (C.57). Furthermore, provided $\theta(\underline{a}) < 0$ and $c'_{1,0}(a) > 0$, it follows that $\tau\left(\underline{a}\right) < 0$ and thus

$$\eta_{3,1}(\underline{a}) = \eta_{3,0}(\underline{a})\tau\left(\underline{a}\right) > 0.$$

As a result, it holds that $v_{1,1} = \sum_{i=1}^{3} \eta_{i,1}\left(\underline{a}\right) > 0$. □

Appendix E. Proof of Lemma 3.5

Proof. Supposing equation (3.25) holds, it follows that

$$v'_j\left(a\right) \approx v'_{j,0}\left(a\right) + \varepsilon v'_{j,1}\left(a\right), \; v''_j\left(a\right) \approx v''_{j,0}\left(a\right) + \varepsilon v''_{j,1}\left(a\right). \tag{E.61}$$

Therefore, by (A.34), we have

$$\begin{aligned} c_j(a) &= v_j'(a)^{-\frac{1}{\gamma}} \approx \left(v_{j,0}'(a)+\varepsilon v_{j,1}'(a)\right)^{-\frac{1}{\gamma}} \\ &= v_{j,0}'(a)^{-\frac{1}{\gamma}}\left(1+\varepsilon\frac{v_{j,1}'(a)}{v_{j,0}'(a)}\right)^{-\frac{1}{\gamma}} \\ &\approx v_{j,0}'(a)^{-\frac{1}{\gamma}}\left(1-\varepsilon\frac{1}{\gamma}\frac{v_{j,1}'(a)}{v_{j,0}'(a)}\right) \\ &= c_{j,0}(a)+\varepsilon c_{j,1}(a), \end{aligned} \tag{E.62}$$

where

$$c_{j,0}(a) = v_{j,0}'(a)^{-\frac{1}{\gamma}},\ c_{j,1}(a) = -\frac{1}{\gamma}v_{j,0}'(a)^{-\frac{1}{\gamma}-1}v_{j,1}'(a). \tag{E.63}$$

Equations (A.35) and (A.36) imply that

$$\begin{aligned} k_j(a) &= \frac{R-r}{\sigma^2}\frac{v_j'(a)}{\varepsilon v_j'(a)^2-v_j''(a)} \\ &\approx \frac{R-r}{\sigma^2}\left(-\frac{v_j'(a)}{v_j''(a)}-\varepsilon\frac{v_j'(a)^3}{v_j''(a)^2}\right) \\ &\approx \frac{R-r}{\sigma^2}\left(-\frac{v_{j,0}'(a)+\varepsilon v_{j,1}'(a)}{v_{j,0}''(a)+\varepsilon v_{j,1}''(a)}-\varepsilon\frac{\left(v_{j,0}'(a)+\varepsilon v_{j,1}'(a)\right)^3}{\left(v_{j,0}''(a)+\varepsilon v_{j,1}''(a)\right)^2}\right) \\ &\approx k_{j,0}(a)+\varepsilon k_{j,1}(a), \end{aligned} \tag{E.64}$$

where

$$k_{j,0}(a) = -\frac{R-r}{\sigma^2}\frac{v_{j,0}'(a)}{v_{j,0}''(a)},\ k_{j,1}(a) = -\frac{R-r}{\sigma^2}\frac{v_{j,1}'(a)\,v_{j,0}''(a)+v_{j,0}'(a)\left(v_{j,0}'(a)^2-v_{j,1}''(a)\right)}{v_{j,0}''(a)^2}. \tag{E.65}$$

Similarly, we have

$$\begin{aligned} h_j(a) &= -\frac{R-r}{\sigma}\frac{\varepsilon v_j'(a)^2}{\varepsilon v_j'(a)^2-v_j''(a)} \\ &\approx -\frac{R-r}{\sigma}\frac{\varepsilon\left(v_{j,0}'(a)+\varepsilon v_{j,1}'(a)\right)^2}{\varepsilon\left(v_{j,0}'(a)+\varepsilon v_{j,1}'(a)\right)^2-\left(v_{j,0}''(a)+\varepsilon v_{j,1}''(a)\right)} \\ &\approx \varepsilon h_{j,1}(a), \end{aligned} \tag{E.66}$$

where

$$h_{j,1}(a) = \frac{R-r}{\sigma}\frac{v'_{j,0}(a)^2}{v''_{j,0}(a)}. \tag{E.67}$$

Substituting (3.25), (E.61), (E.62), (E.64) and (E.66) into both sides of equation (2.11), and collecting terms with the same ε power yield that $v_{j,0}(a)$ and $v_{j,1}(a)$ satisfy the following two coupled functional equations:

$$\begin{aligned} \rho v_{j,0}(a) &= \frac{c_{j,0}(a)^{1-\gamma}}{1-\gamma} + v'_{j,0}(a)\left(z_j + ra + k_{j,0}(a)(R-r) - c_{j,0}(a)\right) && \text{(E.68)} \\ &+\frac{1}{2}\sigma^2 v''_{j,0}(a)\, k^2_{j,0}(a) + \lambda_j\left(v_{-j,0}(a) - v_{j,0}(a)\right), \\ \rho v_{j,1}(a) &= c_{j,0}(a)^{-\gamma} c_{j,1}(a) + v'_{j,0}(a)\left(k_{j,1}(a)(R-r) - c_{j,1}(a)\right) && \text{(E.69)} \\ &+v'_{j,1}(a)\left(z_j + ra + k_{j,0}(a)(R-r) - c_{j,0}(a)\right) \\ &+\sigma^2\left(v''_{j,0}(a)\, k_{j,0}(a)\, k_{j,1}(a) + \frac{1}{2} v''_{j,1}(a)\, k^2_{j,0}(a)\right) \\ &-\frac{1}{2}\sigma^2 v'_{j,0}(a)^2\, k_{j,0}(a)^2 + \lambda_j\left(v_{-j,1}(a) - v_{j,1}(a)\right). \end{aligned}$$

Next, we prove the homogeneity results. By (3.26), we have

$$v_{j,0}(a) = \xi^{1-\gamma} v_{\xi,j,0}\left(\frac{a}{\xi}\right),\ v_{j,1}(a) = \xi^{2(1-\gamma)} v_{\xi,j,1}\left(\frac{a}{\xi}\right),$$

and thus

$$\begin{aligned} v'_{j,0}(a) &= \xi^{-\gamma} v'_{\xi,j,0}\left(\frac{a}{\xi}\right),\ v'_{j,1}(a) = \xi^{1-2\gamma} v'_{\xi,j,1}\left(\frac{a}{\xi}\right), \\ v'_{j,0}(a) &= \xi^{-\gamma-1} v'_{\xi,j,0}\left(\frac{a}{\xi}\right),\ v''_{j,1}(a) = \xi^{-2\gamma} v_{\xi,j,1}\left(\frac{a}{\xi}\right). \end{aligned}$$

It then follows from (E.63), (E.65) and (E.67) that

$$c_{j,0}(a) = v'_{j,0}(a)^{-\frac{1}{\gamma}} = \left(\xi^{-\gamma} v'_{\xi,j,0}\left(\frac{a}{\xi}\right)\right)^{-\frac{1}{\gamma}} = \xi\left(v'_{\xi,j,0}\left(\frac{a}{\xi}\right)\right)^{-\frac{1}{\gamma}} = \xi c_{\xi,j,0}\left(\frac{a}{\xi}\right) \tag{E.70}$$

$$\begin{aligned} c_{j,1}(a) &= -\frac{1}{\gamma} v'_{j,0}(a)^{-\frac{1}{\gamma}-1} v'_{j,1}(a) = -\frac{1}{\gamma}\left(\xi^{-\gamma} v'_{\xi,j,0}\left(\frac{a}{\xi}\right)\right)^{-\frac{1}{\gamma}-1} \xi^{1-2\gamma} v'_{\xi,j,1}\left(\frac{a}{\xi}\right) \\ &= -\xi^{2-\gamma}\frac{1}{\gamma} v'_{\xi,j,0}\left(\frac{a}{\xi}\right)^{-\frac{1}{\gamma}-1} v'_{\xi,j,1}\left(\frac{a}{\xi}\right) = \xi^{2-\gamma} c_{\xi,j,1}\left(\frac{a}{\xi}\right) \end{aligned} \tag{E.71}$$

$$k_{j,0}(a) = -\frac{R-r}{\sigma^2}\frac{v'_{j,0}(a)}{v''_{j,0}(a)} = -\frac{R-r}{\sigma^2}\frac{\xi^{-\gamma} v'_{\xi,j,0}\left(\frac{a}{\xi}\right)}{\xi^{-\gamma-1} v'_{\xi,j,0}\left(\frac{a}{\xi}\right)} = \xi k_{\xi,j,0}\left(\frac{a}{\xi}\right) \tag{E.72}$$

$$k_{j,1}(a) = -\frac{R-r}{\sigma^2}\frac{v'_{j,1}(a)\, v''_{j,0}(a) + v'_{j,0}(a)\left(v'_{j,0}(a)^2 - v''_{j,1}(a)\right)}{\left(v''_{j,0}(a)\right)^2} = \xi^{2-\gamma} k_{\xi,j,1}\left(\frac{a}{\xi}\right) \tag{E.73}$$

$$h_{j,1}(a) = \frac{R-r}{\sigma}\frac{v'_{j,0}(a)^2}{v''_{j,0}(a)} = \frac{R-r}{\sigma}\frac{\left(\xi^{-\gamma} v'_{\xi,j,0}\left(\frac{a}{\xi}\right)\right)^2}{\xi^{-2\gamma} v_{\xi,j,1}\left(\frac{a}{\xi}\right)} = \xi^{1-\gamma} h_{\xi,j,1}\left(\frac{a}{\xi}\right). \tag{E.74}$$

By plugging equations (E.70) to (E.74) into (E.68) and (E.69) and rearranging terms, we obtain that $v_{\xi,j,0}(a)$ and $v_{\xi,j,1}(a)$ satisfy the following two coupled functional equations:

$$\begin{aligned} \rho v_{\xi,j,0}(a) &= \frac{c_{\xi,j,0}(a)^{1-\gamma}}{1-\gamma} + v'_{\xi,j,0}(a)\left(\frac{z_j}{\xi} + ra + k_{\xi,j,0}(a)(R-r) - c_{\xi,j,0}(a)\right) \\ &\quad + \frac{1}{2}\sigma^2 v''_{\xi,j,0}(a)\, k^2_{\xi,j,0}(a) + \lambda_j\left(v_{\xi,-j,0}(a) - v_{\xi,j,0}(a)\right), \end{aligned} \tag{E.75}$$

$$\begin{aligned} \rho v_{\xi,j,1}(a) &= c_{\xi,j,0}(a)^{-\gamma} c_{\xi,j,1}(a) + v'_{\xi,j,0}(a)\left(k_{\xi,j,1}(a)(R-r) - c_{\xi,j,1}(a)\right) \\ &\quad + v'_{\xi,j,1}(a)\left(\frac{z_j}{\xi} + ra + k_{\xi,j,0}(a)(R-r) - c_{\xi,j,0}(a)\right) \\ &\quad + \sigma^2\left(v''_{\xi,j,0}(a)\, k_{\xi,j,0}(a)\, k_{\xi,j,1}(a) + \frac{1}{2} v''_{\xi,j,1}(a)\, k^2_{\xi,j,0}(a)\right) \\ &\quad - \frac{1}{2}\sigma^2 v'_{\xi,j,0}(a)^2\, k_{\xi,j,0}(a)^2 + \lambda_j\left(v_{\xi,-j,1}(a) - v_{\xi,j,1}(a)\right), \end{aligned} \tag{E.76}$$

where in (E.75) and (E.76) we define

$$c_{\xi,j,0}(a) = v'_{\xi,j,0}(a)^{-\frac{1}{\gamma}}, \; c_{\xi,j,1}(a) = -\frac{1}{\gamma} v'_{\xi,j,0}(a)^{-\frac{1}{\gamma}-1} v'_{\xi,j,1}(a), \tag{E.77}$$

$$k_{\xi,j,0}(a) = -\frac{R-r}{\sigma^2}\frac{v'_{\xi,j,0}(a)}{v''_{\xi,j,0}(a)}, \; k_{\xi,j,0}(a) = -\frac{R-r}{\sigma^2}\frac{\begin{array}{c} v'_{\xi,j,1}(a)\, v''_{\xi,j,0}(a) \\ +v'_{\xi,j,0}(a)\left(v'_{\xi,j,0}(a)^2 - v''_{\xi,j,1}(a)\right)\end{array}}{v''_{\xi,j,0}(a)^2}. \tag{E.78}$$

□

Appendix F. Proof of Proposition 3.6

Proof. By Lemma 3.5, we have for any $a \in (\underline{a}, \infty)$,

$$\lim_{\xi\to\infty} v_{\xi,j,0}(a) = \tilde{v}_0(a), \; \lim_{\xi\to\infty} v_{\xi,j,1}(a) = \tilde{v}_1(a), \tag{F.79}$$

where $\tilde{v}_0(a)$ and $\tilde{v}_1(a)$ solve the following two functional equations

$$\rho\tilde{v}_0(a) = \frac{\tilde{c}_0(a)^{1-\gamma}}{1-\gamma} + \tilde{v}'_0(a)\left(ra + \tilde{k}_0(a)(R-r) - \tilde{c}_0(a)\right) + \frac{1}{2}\sigma^2\tilde{v}''_0(a)\,\tilde{k}_0^2(a) \tag{F.80}$$

$$\begin{aligned} \rho\tilde{v}_1(a) &= \tilde{c}_0(a)^{-\gamma}\,\tilde{c}_1(a) + \tilde{v}'_0(a)\left(\tilde{k}_1(a)(R-r) - \tilde{c}_1(a)\right) \\ &\quad +\tilde{v}'_1(a)\left(ra + \tilde{k}_0(a)(R-r) - \tilde{c}_0(a)\right) \\ &\quad +\sigma^2\left(\tilde{v}''_0(a)\,\tilde{k}_0(a)\,\tilde{k}_1(a) + \frac{1}{2}\tilde{v}''_1(a)\,\tilde{k}_0^2(a)\right) - \frac{1}{2}\sigma^2\tilde{v}'_0(a)^2\,\tilde{k}_0(a)^2, \end{aligned} \tag{F.81}$$

with

$$\tilde{c}_0(a) = \tilde{v}'_0(a)^{-\frac{1}{\gamma}}, \; \tilde{c}_1(a) = -\frac{1}{\gamma}\tilde{v}'_0(a)^{-\frac{1}{\gamma}-1}\,\tilde{v}'_1(a) \tag{F.82}$$

$$\tilde{k}_0(a) = -\frac{R-r}{\sigma^2}\frac{\tilde{v}'_0(a)}{\tilde{v}''_0(a)}, \; \tilde{k}_1(a) = -\frac{R-r}{\sigma^2}\frac{\tilde{v}'_1(a)\,\tilde{v}''_0(a) + \tilde{v}'_0(a)\left(\tilde{v}'_0(a)^2 - \tilde{v}''_1(a)\right)}{\tilde{v}''_0(a)^2} \tag{F.83}$$

$$\tilde{h}_1(a) = \frac{R-r}{\sigma}\frac{\tilde{v}'_0(a)^2}{\tilde{v}''_0(a)}. \tag{F.84}$$

It is straightforward to verify that $\tilde{v}(a) = \tilde{v}_1(a) + \varepsilon\tilde{v}_0(a)$ is an approximate solution to the HJB equation (2.11) without income uncertainty and borrowing constraint. Combining equation (E.70) with (F.79) leads to for a very large a,

$$c_{j,0}(a) = \xi c_{\xi,j,0}\left(\frac{a}{\xi}\right) = a c_{a,j,0}(1) \approx a\tilde{c}_0(1). \tag{F.85}$$

Similarly, we have

$$c_{j,1}(a) = \xi^{2-\gamma} c_{\xi,j,1}\left(\frac{a}{\xi}\right) = a^{2-\gamma} c_{a,j,1}(1) \approx a^{2-\gamma}\tilde{c}_1(1) \tag{F.86}$$

$$k_{j,0}(a) = \xi k_{\xi,j,0}\left(\frac{a}{\xi}\right) = \xi k_{a,j,0}(1) \approx a\tilde{k}_0(1) \tag{F.87}$$

$$k_{j,1}(a) = \xi^{2-\gamma} k_{\xi,j,1}\left(\frac{a}{\xi}\right) = a^{2-\gamma} k_{a,j,1}(1) \approx a^{2-\gamma}\tilde{k}_1(1) \tag{F.88}$$

$$h_{j,1}(a) = \xi^{1-\gamma} h_{\xi,j,1}\left(\frac{a}{\xi}\right) = a^{1-\gamma} h_{a,j,1}(1) \approx a^{1-\gamma}\tilde{h}_1(1). \tag{F.89}$$

Therefore, it remains to solve equations (F.80) and (F.81). First, conjecture

$$\tilde{v}_0(a) = \alpha_0 \frac{a^{1-\gamma}}{1-\gamma} \tag{F.90}$$

for some number α_0. Then we have $\tilde{v}_0'(a) = \alpha_0 a^{-\gamma}$, $\tilde{v}_0''(a) = -\gamma\alpha_0 a^{-\gamma-1}$, and subsequently from (F.82) to (F.84) that

$$\tilde{c}_0(a) = \alpha_0^{-\frac{1}{\gamma}} a,\ \tilde{k}_0(a) = \frac{R-r}{\sigma^2} a,\ \tilde{h}_1(a) = -\frac{R-r}{\sigma^2}\alpha_0 a^{1-\gamma}. \tag{F.91}$$

Substituting (F.90) and (F.91) into (F.80) yields the value of α_0 given by (3.30). Second, conjecture

$$\tilde{v}_1(a) = \alpha_1 \frac{a^{2(1-\gamma)}}{2(1-\gamma)} \tag{F.92}$$

for some number α_1. Then we have $\tilde{v}_1'(a) = \alpha_1 a^{1-2\gamma}$, $\tilde{v}_1''(a) = (1-2\gamma)\alpha_1 a^{1-2\gamma}$, and subsequently from (F.82) to (F.84) that

$$\tilde{c}_1(a) = -\frac{1}{\gamma}\alpha_0^{-\frac{1}{\gamma}-1}\alpha_1 a^{2-\gamma},\ \tilde{k}_1(a) = -\frac{R-r}{\sigma^2}\frac{\alpha_0^2 + (\gamma-1)\alpha_1}{\gamma^2\alpha_0} a^{2-\gamma}. \tag{F.93}$$

Plugging (F.92) and (F.93) into (F.81) results in

$$\alpha_1\left(r - \alpha_0^{-\frac{1}{\gamma}} + \frac{1}{2}\frac{(R-r)^2}{\gamma^2\sigma^2} - \frac{\rho}{2(1-\gamma)}\right) = -\frac{1}{2}\alpha_0^2\frac{(R-r)^2}{\gamma^2\sigma^2},$$

and thus the value of α_1 as in (3.31). By (E.62), (F.85) and (F.86), when a is sufficiently large, we have

$$c_j\left(a\right) \approx c_{j,0}\left(a\right) + \varepsilon c_{j,1}\left(a\right) \approx a\tilde{c}_0\left(1\right) + \varepsilon a^{2-\gamma}\tilde{c}_1\left(1\right) = \alpha_0^{-\frac{1}{\gamma}} a - \varepsilon \frac{1}{\gamma} \alpha_0^{-\frac{1}{\gamma}-1} \alpha_1 a^{2-\gamma}.$$

By the same token,

$$k_j\left(a\right) \approx k_{j,0}\left(a\right) + \varepsilon k_{j,1}\left(a\right) \approx a\tilde{k}_0\left(1\right) + \varepsilon a^{2-\gamma}\tilde{k}_1\left(1\right) = \frac{R-r}{\sigma^2} a - \varepsilon \frac{R-r}{\sigma^2} \frac{\alpha_0^2 + \left(\gamma - 1\right)\alpha_1}{\gamma^2 \alpha_0} a^{2-\gamma}$$

$$h_j\left(a\right) \approx \varepsilon h_{j,1}\left(a\right) \approx \varepsilon a^{1-\gamma}\tilde{h}_1\left(1\right) = -\varepsilon \frac{R-r}{\sigma^2} \alpha_0 a^{1-\gamma}.$$

□

Appendix G. Proof of Proposition 4.1

Proof. First of all, note the aggregate saving function $S(r) = \sum_{j=1}^{2} \int_{\underline{a}}^{\infty} \left(a - k_j(a)\right) g_j(a) da$ is continuous in r. This is because both individual investment functions $k_j\left(a\right)$ and the stationary density functions $g_j\left(a\right)$ are continuous in r.

Next, we show $\lim_{r\to-\infty} S(r) = \underline{a}$. Indeed, by the optimality condition for k in (A.37), when $r \to -\infty$,

$$k_j(a) = \min\left\{\frac{R-r}{\sigma^2} \frac{v_j'(a)}{\varepsilon v_j'^2 - v_j''(a)}; a + \phi\right\} = a + \phi.$$

By construction,

$$b_j\left(a\right) = a - k_j\left(a\right) = -\phi = \underline{a},$$

which implies that all households will borrow up to the limit and use the acquired resources to purchase the risky asset. Consequently,

$$\lim_{r\to-\infty} S(r) = \lim_{r\to-\infty} \sum_{j=1}^{2} \int_{\bar{a}}^{\infty} b_j(a) g_j(a) da = \underline{a} < 0. \tag{G.94}$$

As the third step, we show $\lim_{r\to\bar{r}} S(r) = \infty$, where $\bar{r} = \max\left\{R, \rho\right\}$. First, consider the case with $R \leq \rho$. By (A.37), we have for $r \geq R$,

$$k_j(a) = \frac{R-r}{\sigma^2} \frac{v_j'(a)}{\varepsilon v_j'^2 - v_j''(a)} \leq 0.$$

Since $k_j(a)$ is non-negative, $k_j(a) = 0$ for all a. In other words, when the risk-free rate exceeds the gross return on the risky asset, households would not invest in the asset at all. In this case, they will behave as if they were in a problem without a risky asset. The corresponding Euler equation governing the consumption-saving behavior of households in the high-income state is

$$\frac{u''(c_2(a))}{u'(c_2(a))} c_2'(a)\, s_2(a) = \rho - r - \lambda_2 \left(\frac{u'(c_1(a))}{u'(c_2(a))} - 1 \right).$$

Since $c_2(a) > c_1(a)$, the right-hand side of the above equation becomes strictly negative when r approaches ρ from below. This implies that $\lim_{r \uparrow \rho} s_2(a) > 0$, suggesting that households in the high-income state accumulate the risk-free asset, leading to

$$\lim_{r \uparrow \rho} S(r) = \infty. \tag{G.95}$$

Now, consider the case with $R > \rho$. We know when r approaches R from below, risk-averse households will allocate their wealth towards the risk-free bond, and in the limit, they hold zero risky assets. Since $R > \rho$, as $r \uparrow R$, households will save as much as they can in the risk-free bond, and thus

$$\lim_{r \uparrow R} S(r) = \infty. \tag{G.96}$$

Given equations (G.94), (G.95) and (G.96), the Intermediate Value Theorem implies that there exists an interest rate $r^* \in (-\infty, \bar{r})$ such that $S(r^*) = 0$. $\square$